* The temporal Heisenberg inequality

- Ehrenfest theorem:

$$\frac{d}{dt}\langle A\rangle_\varphi = \frac{1}{i\hbar}\langle [A,H]\rangle_\varphi \qquad \| \ \langle\cdot\rangle_\varphi = \langle\varphi(t)|\cdot|\varphi(t)\rangle$$

- uncertainty relation : $\langle(\Delta A)^2\rangle_\varphi \langle(\Delta B)^2\rangle_\varphi \geq \frac{1}{4}|\langle[A,B]\rangle_\varphi|^2$

$\langle(\Delta A)^2\rangle_\varphi \equiv (\Delta_\varphi A)^2$.

Let's put H into B!

$$\rightarrow \quad \Delta_\varphi H\,\Delta_\varphi A \geq \frac{1}{2}|\langle[A,H]\rangle_\varphi| = \frac{1}{2}\hbar\left|\frac{d}{dt}\langle A\rangle_\varphi\right|$$

If we define the time $\tau_\varphi(A)$ as

$$\frac{1}{\tau_\varphi(A)} \equiv \left|\frac{d\langle A\rangle_\varphi}{dt}\right|\frac{1}{\Delta_\varphi A},$$

then τ_φ = characteristic time for the expectation value of A to change by $\Delta_\varphi A$.

$$\Rightarrow \quad \Delta_\varphi H\,\tau_\varphi(A) \geq \frac{1}{2}\hbar \quad \Rightarrow \quad \Delta E\,\Delta t \gtrsim \frac{1}{2}\hbar \quad //$$

Energy spread — characteristic evolution time.

2.3 Simple Harmonic oscillator

(1) Energy eigenkets. (Dirac's operator method)

$$H = \frac{\tilde{p}^2}{2m} + \frac{1}{2}m\omega^2\tilde{x}^2 = \hbar\omega\left(\tilde{a}^+\tilde{a} + \frac{1}{2}\right)$$

$$\equiv \hbar\omega\left(\tilde{N} + \frac{1}{2}\right)$$

def.

annihilation operator

$$\tilde{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\tilde{x} + i\frac{\tilde{p}}{m\omega}\right)$$

creation operator

$$\tilde{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(\tilde{x} - i\frac{\tilde{p}}{m\omega}\right)$$

$$\Rightarrow \quad \tilde{x} = \frac{x_0}{\sqrt{2}}(\tilde{a} + \tilde{a}^+), \qquad \tilde{p} = i\frac{\hbar}{\sqrt{2}x_0}(-\tilde{a} + \tilde{a}^+)$$

$$\| \ x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

$$\tilde{N} = \tilde{a}^+\tilde{a}$$

→ Commutation relation $[\tilde{a}, \tilde{a}^+] = 1$

$\Rightarrow [H, \tilde{N}] = 0$; There's simultaneous eigenkets

$$\therefore \quad H|n\rangle = (n+\tfrac{1}{2})\hbar\omega|n\rangle \quad \| \quad \tilde{N}|n\rangle = n|n\rangle$$

same eigenkets $\{|n\rangle\}$.

$$\Rightarrow \boxed{E_n = (n+\tfrac{1}{2})\hbar\omega} \quad \| \quad H|n\rangle = E_n|n\rangle$$

Q. What's "n", then? We know that $n = 0, 1, 2, \cdots$

But, how do we know that from what we have?

Let's look at the commutation relation:

$$[\tilde{N}, \tilde{a}] = [\tilde{a}^+\tilde{a}, \tilde{a}] = \tilde{a}^+\underbrace{[\tilde{a}, \tilde{a}]}_{0} + \underbrace{[\tilde{a}^+, \tilde{a}]}_{-1}\tilde{a}$$

$$= -\tilde{a}$$

likewise, $[\tilde{N}, \tilde{a}^+] = \tilde{a}^+$.

Now, try $\tilde{N}\tilde{a}^+|n\rangle = ([N, \tilde{a}^+] + \tilde{a}^+\tilde{N})|n\rangle$

switch!

$$= \tilde{a}^+(\tilde{N}+1)|n\rangle$$

$$\Rightarrow \quad \tilde{N}\tilde{a}^+|n\rangle = (n+1)\tilde{a}^+|n\rangle$$

Thus, $\tilde{a}^+$ is a creation operator as it makes $n \to n+1$

likewise, $\tilde{N}\tilde{a}|n\rangle = ([\tilde{N}, \tilde{a}] + \tilde{a}\tilde{N})|n\rangle$

$$= (n-1)|n\rangle$$

$\Rightarrow$ $\tilde{a}$ is an annihilation operator.

$(n \to n-1)$

$$\left[\begin{array}{l}\tilde{N}\tilde{a}^+|n\rangle = (n+1)\tilde{a}^+|n\rangle \\ \tilde{N}\tilde{a}|n\rangle = (n-1)\tilde{a}|n\rangle\end{array}\right. \quad \xrightarrow{\text{implies}} \quad \boxed{\begin{array}{l}\tilde{a}^+|n\rangle = C_+|n+1\rangle \\ \tilde{a}|n\rangle = C_-|n-1\rangle\end{array}}$$

$C_\pm$: c-number.

* Let's check : $\tilde{N}\tilde{a}^+|n\rangle = (n+1)\tilde{a}^+|n\rangle$

$$\Rightarrow \quad \tilde{N}\cdot \cancel{C_+}|n+1\rangle = (n+1)\cdot\cancel{C_+}\cdot|n+1\rangle$$

$$\tilde{N}|n+1\rangle = (n+1)|n+1\rangle \quad ; \text{ OK!}$$

Now, Let's determine $C_\pm$.

To recover $\tilde{N}|n\rangle = n|n\rangle$,

$$\langle n|\tilde{a}^+\tilde{a}|n\rangle = |C_-|^2\langle n-1|n-1\rangle = n$$

$$\therefore \quad \underline{C_- = \sqrt{n}} \quad //$$

choose $C_\pm$ to be real and positive.

likewise, $\langle n|\tilde{a}^+\tilde{a}|n\rangle = \langle n|\cdot\left(\tilde{a}\tilde{a}^+ \underbrace{- \tilde{a}\tilde{a}^+ + \tilde{a}^+\tilde{a}}_{=[\tilde{a}^+,\tilde{a}]=-1}\right)\cdot|n\rangle$

$$= \langle n|\tilde{a}\tilde{a}^+|n\rangle - 1$$

$$\Rightarrow \quad |C_+|^2 = n+1 \quad \therefore \quad \underline{C_+ = \sqrt{n+1}} \quad //$$

Now, we know that, if $(n, |n\rangle)$ is the eigenpair,

So ARE. $\{\cdots (n-2, |n-2\rangle), (n-1, |n-1\rangle),$
$(n+1, |n+1\rangle), (n+2, |n+2\rangle), \cdots\}$.

; Not enough to determine n.
Is there any lower bound ?

$\boxed{\therefore\ n = 0, 1, 2, 3, 4, \cdots}$

$$\Rightarrow \quad \langle n|\tilde{N}|n\rangle = (\langle n|\tilde{a}^+)\cdot(a|n\rangle) \geq 0.$$

$$\therefore \quad \underline{n \geq 0}$$

$\Rightarrow$ Ground-state energy , Eigen state

$$E_0 = \frac{1}{2}\hbar\omega \quad \longleftarrow \quad |0\rangle$$

$\hookrightarrow |1\rangle$

$\hookrightarrow |2\rangle$

$\hookrightarrow \vdots$

$$|n+1\rangle = \frac{\tilde{a}^+}{\sqrt{n+1}}|n\rangle$$

$\Rightarrow$ n-th excited state

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega \quad \longleftarrow \quad \left[\frac{(\tilde{a}^+)^n}{\sqrt{n!}}\right]|0\rangle \equiv |n\rangle .$$

$n = 0, 1, 2, \cdots$

* matrix representation in the basis of $\{|n\rangle\}$.

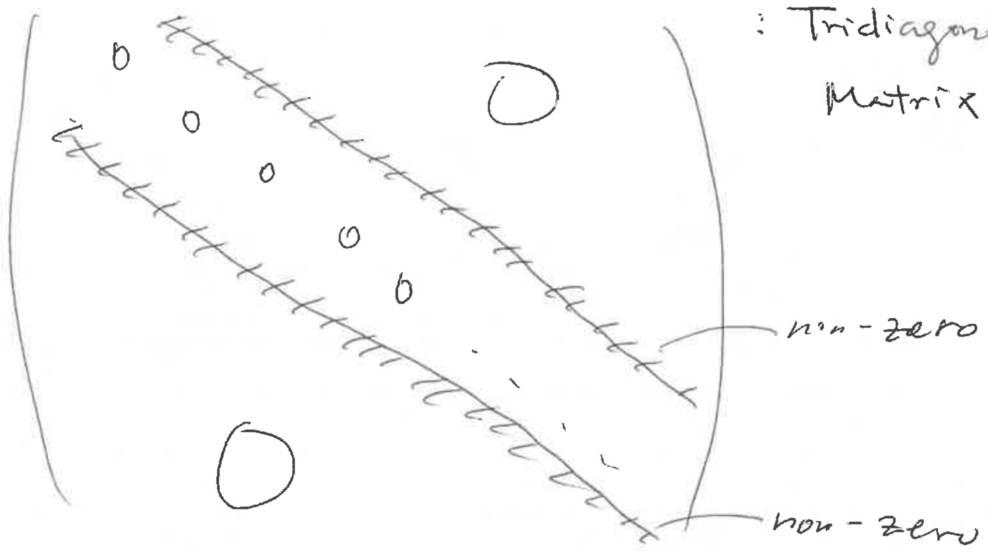
- $\langle n'|\tilde{a}|n\rangle = \sqrt{n}\,\delta_{n',n-1}$, $\langle n'|\tilde{a}^+|n\rangle = \sqrt{n+1}\,\delta_{n',n+1}$

- $\tilde{x}$ and $\tilde{p}$?

$$\begin{cases} \tilde{x} = \sqrt{\dfrac{\hbar}{2m\omega}}\,(\tilde{a} + \tilde{a}^+) \\[2ex] \tilde{p} = i\sqrt{\dfrac{m\hbar\omega}{2}}\,(-\tilde{a} + \tilde{a}^+) \end{cases}$$

$$\Rightarrow \langle n'|\tilde{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\delta_{n',n-1} + \sqrt{n+1}\,\delta_{n',n+1}\right).$$

$$\langle n'|\tilde{p}|n\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(-\sqrt{n}\,\delta_{n',n-1} + \sqrt{n+1}\,\delta_{n',n+1}\right).$$

$\Rightarrow$ 

: Tridiagonal Matrix.

- Energy eigenfunction in x-space.

  - ground-state wave function $\langle x|0\rangle$:

    Ground-state ket $|0\rangle$ statisfies

$$\tilde{a}|0\rangle = 0 \Rightarrow \langle x|\tilde{a}|0\rangle = \sqrt{\frac{m\omega}{2\hbar}}\langle x|\,\tilde{x} + i\frac{\tilde{p}}{m\omega}|0\rangle$$

$$\Rightarrow \left(x + x_0^2\frac{\partial}{\partial x}\right)\langle x|0\rangle = 0 \qquad \Big\| \; x_0^2 = \frac{\hbar}{m\omega}$$

1st. order diff. eq.

$$\langle x|0\rangle = A e^{-\frac{1}{2}\frac{x^2}{x_0^2}} \quad \oplus \text{ normalization } \int_{-\infty}^{\infty} dx \, \langle x|0\rangle = 1$$

$$= \frac{1}{\pi^{1/4}\sqrt{x_0}} \exp\left[-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right].$$

  - excited states.

$$\langle x|1\rangle = \langle x|\tilde{a}^{+}|0\rangle = \sqrt{\frac{m\omega}{2\hbar}}\langle x|\,\tilde{x} - i\frac{\tilde{p}}{m\omega}|0\rangle$$

$$= \frac{1}{\sqrt{2}x_0}\left(x - x_0^2\frac{\partial}{\partial x}\right)\langle x|0\rangle$$

$\vdots$

$$\langle x|n\rangle = \frac{1}{\sqrt{n!}}\langle x|(\tilde{a}^{+})^n|0\rangle = \frac{1}{\sqrt{n!}}\left(\frac{1}{\sqrt{2}x_0}\right)^n\left(x - x_0^2\frac{\partial}{\partial x}\right)^n\langle x|0\rangle$$

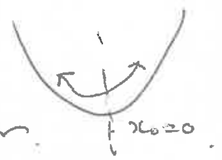
$$= \frac{1}{\pi^{1/4}\sqrt{2^n n!}}\left(\frac{1}{x_0^{n+\frac{1}{2}}}\right)\left(x - x_0^2\frac{\partial}{\partial x}\right)^n \exp\left(-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right).$$

- Let's check up what we know from the uncertainty principle. $\Delta x \Delta p \gtrsim \frac{\hbar}{2}$

  - classical Hamiltonian :

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = E \quad \text{(Energy)}.$$

we know $\Delta p = p$, $\Delta x = x$ since it's an oscillator.



$$E = \frac{\Delta p^2}{2m} + \frac{1}{2} m\omega^2 \Delta x^2 \geq \underbrace{\frac{\hbar^2}{8m}\Delta x^{-2} + \frac{1}{2} m\omega^2 \Delta x^2}$$

has minimum at $\Delta x^2 = \frac{\hbar}{2m\omega}$

$(E_{min} = \frac{\hbar}{2}\omega)$ $\quad \Delta p^2 = \frac{\hbar m\omega}{2}$

$$\geq \frac{1}{4}\hbar\omega + \frac{1}{4}\hbar\omega$$

$$\geq \frac{1}{2}\hbar\omega \equiv E_{ground} \; !$$

- Let's verify these steps with $\langle x|0\rangle$.

$$\langle \tilde{x} \rangle = 0 , \quad \langle \tilde{p} \rangle = 0 .$$

$$\langle \tilde{x}^2 \rangle = \left\langle \frac{\hbar}{2m\omega} (\tilde{a} + \tilde{a}^+)^2 \right\rangle = \frac{\hbar}{2m\omega} \langle 0 | \big(\tilde{a}^2 + \tilde{a}^{+2} + \tilde{a}^+\tilde{a} + \underbrace{\tilde{a}\tilde{a}^+}_{1 + \tilde{a}^+\tilde{a}}\big) | 0 \rangle$$

$$= \frac{\hbar}{2m\omega}$$

Similarly, $\langle \tilde{p}^2 \rangle = \frac{\hbar m\omega}{2}$ $\quad \Leftrightarrow \quad \left(\begin{array}{l} \Delta x^2 = \frac{\hbar}{2m\omega} \\ \Delta p^2 = \frac{\hbar m\omega}{2} \end{array} \right.$

$$\left\langle \frac{\tilde{p}^2}{2m} \right\rangle = \frac{1}{4}\hbar\omega , \quad \left\langle \frac{1}{2} m\omega^2 \tilde{x}^2 \right\rangle = \frac{1}{4}\hbar\omega$$

$$E = \langle H \rangle = \frac{1}{2}\hbar\omega .$$

$$\langle (\Delta\tilde{x})^2 \rangle \langle (\Delta\tilde{p})^2 \rangle = \frac{\hbar}{2m\omega} \cdot \frac{\hbar m\omega}{2} = \frac{\hbar^2}{4}$$

For the nth state,

$$\langle (\Delta\tilde{x})^2 \rangle \langle (\Delta\tilde{p})^2 \rangle = (n + \tfrac{1}{2})^2 \hbar^2 \geq \frac{\hbar^2}{4}$$

(2) Time Development of the Oscillator

— Heisenberg picture "

$$\text{EOM}: \quad \frac{d\tilde{p}(t)}{dt} = \frac{1}{i\hbar}\left[\tilde{p}(t), H\right]$$

$$\frac{d\tilde{x}(t)}{dt} = \frac{1}{i\hbar}\left[\tilde{x}(t), H\right]$$

$$H = \frac{\tilde{p}(t)^2}{2m} + \frac{1}{2}m\omega^2\tilde{x}(t)^2$$

$p(t) \equiv p^{(H)}(t)$
$x(t) \equiv x^{(H)}(t)$

NOTE: We know that $\langle H \rangle$ is conserved, thus, $H(t) = H$ (t-indep.)

But, Let's just try with $H(t)$.

$$\Rightarrow \begin{cases} \dfrac{d\tilde{p}(t)}{dt} = -m\omega^2\tilde{x}(t) \\[2mm] \dfrac{d\tilde{x}(t)}{dt} = \dfrac{p(t)}{m} \end{cases}$$

* Show $[x(t), p(t)] = i\hbar$

when $x(t) = e^{\frac{iH}{\hbar}t}\, x\, e^{-\frac{iH}{\hbar}t}$

$p(t) = e^{\frac{iH}{\hbar}t}\, p\, e^{-\frac{iH}{\hbar}t}$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix}\tilde{x}(t)\\ \tilde{p}(t)\end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m} \\ -m\omega^2 & 0\end{pmatrix}\begin{pmatrix}\tilde{x}(t)\\ \tilde{p}(t)\end{pmatrix}$$

$\equiv A$

diagonalization :

eigenvalues $\begin{cases} i\omega \leftarrow \begin{pmatrix} -\frac{i}{m\omega} \\ 1\end{pmatrix} \\ -i\omega \leftarrow \begin{pmatrix} 1 \\ -im\omega\end{pmatrix}\end{cases}$

$\hookrightarrow$

$$X^{-1}AX = D$$

$$D = \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega\end{pmatrix}$$

$\hookrightarrow$

$$A = XDX^{-1}$$

$$X = \frac{1}{\sqrt{2}}\begin{pmatrix} -\frac{i}{m\omega} & 1 \\ 1 & -im\omega\end{pmatrix}$$

$$\Rightarrow \frac{d}{dt}\left[X^{-1}\begin{pmatrix}\tilde{x}(t)\\ \tilde{p}(t)\end{pmatrix}\right] = D\left[X^{-1}\begin{pmatrix}\tilde{x}(t)\\ \tilde{p}(t)\end{pmatrix}\right]$$

$$X^{-1}\begin{pmatrix}\tilde{x}(t)\\ \tilde{p}(t)\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} i m\omega & 1 \\ 1 & \frac{i}{m\omega}\end{pmatrix}\begin{pmatrix}\tilde{x}(t)\\ \tilde{p}(t)\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} i m\omega\, \tilde{x}(t) + \tilde{p}(t) \\ \tilde{x}(t) + \frac{i}{m\omega}\tilde{p}(t)\end{pmatrix} \equiv A\begin{pmatrix}\hat{\varphi}_1(t)\\ \hat{\varphi}_2(t)\end{pmatrix}$$

→ Any diagonal matrix with c-number, ∴ Choose it to be "I".

$$\Rightarrow \frac{d}{dt}\begin{pmatrix}\hat{\varphi}_1(t)\\ \hat{\varphi}_2(t)\end{pmatrix} = \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega\end{pmatrix}\begin{pmatrix}\hat{\varphi}_1(t)\\ \hat{\varphi}_2(t)\end{pmatrix}.$$

$$\therefore \boxed{\begin{aligned}\hat{\varphi}_1(t) &= c_1 e^{i\omega t}\\ \hat{\varphi}_2(t) &= c_2 e^{-i\omega t}\end{aligned}} \quad \Bigg\| \quad \begin{aligned} c_1 &= \hat{\varphi}_1(0) \\ c_2 &= \hat{\varphi}_2(0)\end{aligned}$$

also for $\tilde{a}^{+}$, $\tilde{a}$

time-invariant : $\varphi_1(t)\cdot\varphi_2(t) = c_1 c_2$. "

$$\left[\tilde{x}(t) - \frac{i}{m\omega}\tilde{p}(t)\right] = \left[\tilde{x}(0) - \frac{i}{m\omega}\tilde{p}(0)\right] e^{i\omega t}$$

$$\left[\tilde{x}(t) + \frac{i}{m\omega}\tilde{p}(t)\right] = \left[\tilde{x}(0) + \frac{i}{m\omega}\tilde{p}(0)\right] e^{-i\omega t}$$

Q. Is it really classical?

$$\Rightarrow \left[\begin{aligned}\tilde{x}(t) &= \tilde{x}(0)\cos\omega t + \frac{\tilde{p}(0)}{m\omega}\sin\omega t \\ \tilde{p}(t) &= -m\omega\,\tilde{x}(0)\sin\omega t + \tilde{p}(0)\cos\omega t\end{aligned}\right. \quad \Bigg\| \quad \begin{aligned}\cos\omega t &= \frac{e^{+i\omega t} + e^{-i\omega t}}{2}\\ \sin\omega t &= \frac{e^{i\omega t} - e^{-i\omega t}}{2i}\end{aligned}$$

☆ What about $\tilde{a}$, $\tilde{a}^{+}$?

Note that. $\hat{\varphi}_1 = \frac{1}{\sqrt{2}}(i m\omega\,\tilde{x} + \tilde{p})$

$\propto \tilde{a}^{+} = \sqrt{\frac{m\omega}{2\hbar}}\left(\tilde{x} - \frac{i\tilde{p}}{m\omega}\right)$.

and $\hat{\varphi}_2 \propto \tilde{a}$; $\Rightarrow \boxed{\left(\begin{aligned}\tilde{a}^{+}(t) &= e^{i\omega t}\tilde{a}^{+}(0)\\ \tilde{a}(t) &= e^{-i\omega t}\tilde{a}(0)\end{aligned}\right.}$

↑ These can be verified by using the Baker-Hausdorff lemma : $A(t) = e^{\frac{iHt}{\hbar}} A e^{-\frac{iHt}{\hbar}}$ $= \cdots$ (pp 95.) of Sakurai.

$\therefore$ $\tilde{a}^{+}(t)\,\tilde{a}(t)$ = time-invariant. ; $[H, \tilde{a}^{+}\tilde{a}] = 0$.

→ simultaneous eigenket !!!

Look at :

$$\left[\begin{aligned} \tilde{x}(t) &= \tilde{x}(0)\cos\omega t + \frac{\tilde{p}(0)}{m\omega}\sin\omega t \\ \tilde{p}(t) &= -m\omega\,\tilde{x}(0)\sin\omega t + \tilde{p}(0)\cos\omega t \end{aligned}\right.$$

$$\& \quad m\frac{d\tilde{x}}{dt} = \tilde{p} \quad , \quad \underline{\text{just like C.M.}}$$

But, $\langle\tilde{x}\rangle$ and $\langle\tilde{p}\rangle$ are "not" oscillating. !!!

although $\tilde{x}(t)$ and $\tilde{p}(t)$ look like oscillating.

|| NOTE : $\langle\tilde{x}\rangle = 0$, $\langle\tilde{p}\rangle = 0$. for all $|n\rangle$.

Q. Can we find a "Quantum" state

that behaves just like classical $\langle x\rangle$ and $\langle p\rangle$?

* Coherent States . ← This is the one.

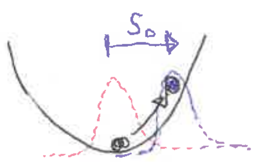
why do we need this ?

- We live in a "classical" world,

But we want to control a "Quantum" world.

: We need a "bridge" !

the easiest way to make the coherent state

→ 

$$|S_0\rangle = J(S_0)\,|0\rangle$$

move the ground state to S_0 .

wave function $\psi_{S_0}(x) = \psi_0(x - S_0)$. $\Big\|$ $\langle x|J(S_0) = \langle x - S_0|$

observables :

$$\langle S_0|\tilde{x}|S_0\rangle = \langle 0|J^\dagger(S_0)\,\tilde{x}\,J(S_0)|0\rangle = S_0$$

$$\langle S_0|\tilde{p}|S_0\rangle = 0 .$$

$$\langle S_0|H|S_0\rangle = \langle 0|\frac{\tilde{p}^2}{2m}|0\rangle + \frac{1}{2}m\omega^2\langle 0|(\tilde{x}+S_0)^2|0\rangle .$$